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History of Mathematics A Paper for IU Mathematics

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Euclid

An Analysis of the Elements

A story has been passed down about Euclid that can possibly help us understand the obsession that was necessary to organize every detail that was known about geometry into a concise and understandable form. It is said that a young boy who was studying Euclid's text on geometry (The Elements) asked him, "But what shall I get by learning these things?" Euclid's answer was that we should all learn for the sake of learning, and it is in this that we profit from our newly found knowledge. (Burton) We continue to profit in mathematics from a text that was written before 300 B.C. in Alexandria , Greece which was a major center of learning at the time. It was here that Euclid had access to the majority of the books that had been written on geometry, as well as mathematical training from the pupils of Plato. The instruction that he received gave Euclid firm background in the use of deductive logic, which was necessary to put together the "elements" of geometry.

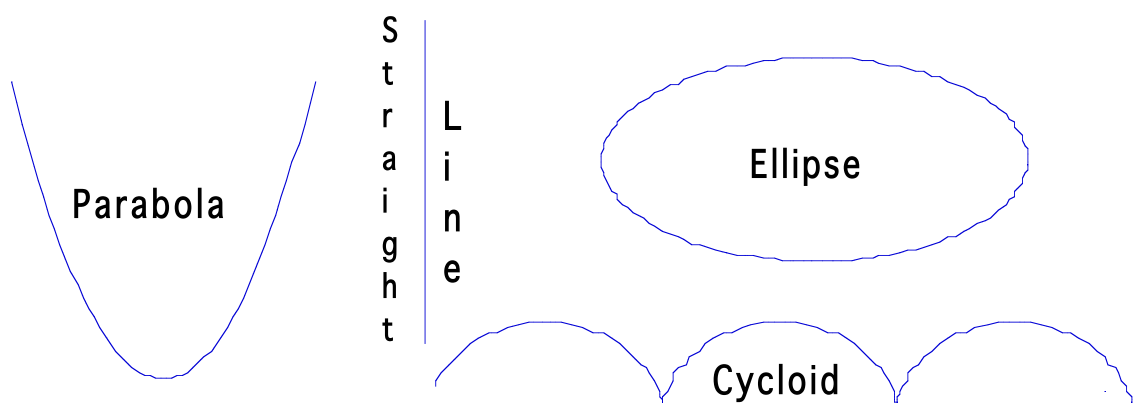
Euclid's Elements represent all of the great achievements of Greek geometry in a well-organized format. The success of the Elements as a text can be compared only to the Bible in terms of circulation. Mathematicians, from Euclid forward, have studied and criticized the Elements in various aspects, but it is from this analysis that we gain a better understanding of geometry. To focus on the inconsistencies within the text itself is not a show of disrespect for Euclid, but it is an attempt to never lose our intuitive desire for knowledge (which Euclid encouraged). And while, according to Euclid, there is no " royal road to geometry " , the Elements have certainly established a nice path for mathematicians to begin their journey.

One of the first mathematicians to embark upon this journey in commenting upon the content of the Elements was Proclus of Alexandria (485 B.C.). It is from Proclus that we know anything at all about the life of Euclid and of Greek mathematics in general. Proclus, like many mathematicians at this time, taught geometry (using the Elements) by analyzing and commenting on Euclid's definitions and propositions. He eventually organized his thoughts from the classroom into a book called "A Commentary on the First Book of Euclid's Elements". In this text, Proclus uses a format in which he first explains

Euclid's proofs, introduces various cases, and then discusses objections from either himself or arguments of other mathematicians.

To begin the book, Proclus makes a general commentary on the Elements as a whole and then goes on to discuss each aspect individually. The first comments about the Elements deal with the definitions of the terms that Euclid uses throughout the entire text. For example, Euclid defines a point as " *That which has no part*" and a line as " *A breadthless length*". Proclus merely offers alternative definitions while admitting that perfect descriptions of these terms did not exist. If an in-depth definition were provided, this might have cause more restrictions later in the Elements. To avoid this problem but still give an alternative definition, Proclus chose to discuss a line in terms of dimensions. His definitions of lines included " A magnitude extended one way " (one-dimensional) and "A flux of a point" (the path of a point when moved).

Having posed these definitions, Proclus then chooses to quote Geminus' discussion of the "Classification of lines" to argue that "A breadthless length" could mean any number of figures. According to Geminus, lines can take two forms: composite and incomposite. Of these, Proclus focuses attention upon the incomposite forms since this includes anything from a 'straight' line to an ellipse. All of the 'incomposite' lines below are an example of a 'breadthless length according to Proclus' argument since all of them could continue toward infinity.



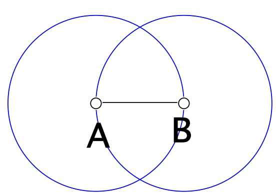
Proclus continued his analysis of each definition as a way of leading into his discussion of Euclid's Propositions, in which much of the emphasis of his criticism is based.

The majority of the criticism that Proclus had about the Propositions mostly dealt with either the construction or the proof of the construction However, for his discussion of Euclid's Proposition I (construction of an equilateral triangle), the argument is not of the importance of the equilateral triangle or of the construction; rather Proclus thought that it would have been appropriate for the constructions of a scalene and isosceles triangles as well. It is obvious that Euclid did not include these constructions because there is no use for them later in the Elements. However, Proclus believed that a geometry text is merely incomplete without the constructions of these two triangles and so argued that these procedures should have been added to the Elements. His proposed constructions are as follows:

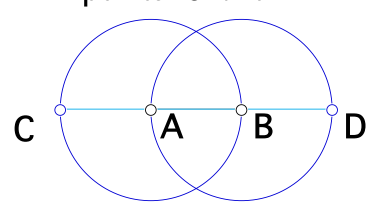
To produce a particular isosceles triangle:

Let Segment AB be the base upon which the isosceles triangle will be constructed.

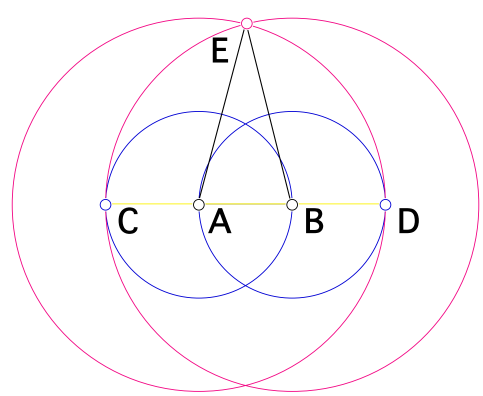
1. Construct a circle around A through B and a circle around B through A as in the equilateral propostion:



2. Extend AB in both directions to form points C and D:



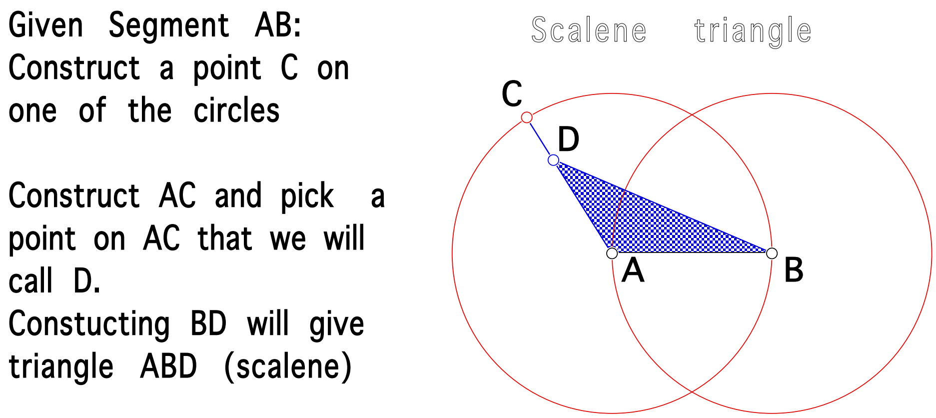
3. Construct a circle around B through point C and also a circle around A through point D. Then, draw segments from point A and B the newly formed point E which forms the isosceles triangle ABE:



(Proclus)

Proclus added that this construction was an elegant method of drawing an isosceles triangle base on the fact that it can yield two different triangles. In the previous construction, we saw that triangle ABE was the end result but we could have just as easily drawn triangle CDE for the isosceles triangle. Many would argue that this only allows for two distinct triangles to be formed and would be of little use to a geometer who needed a particular isosceles triangle.

A similar situation exists for his construction of a scalene triangle since Proclus once again relies upon the first step of the equilateral method:

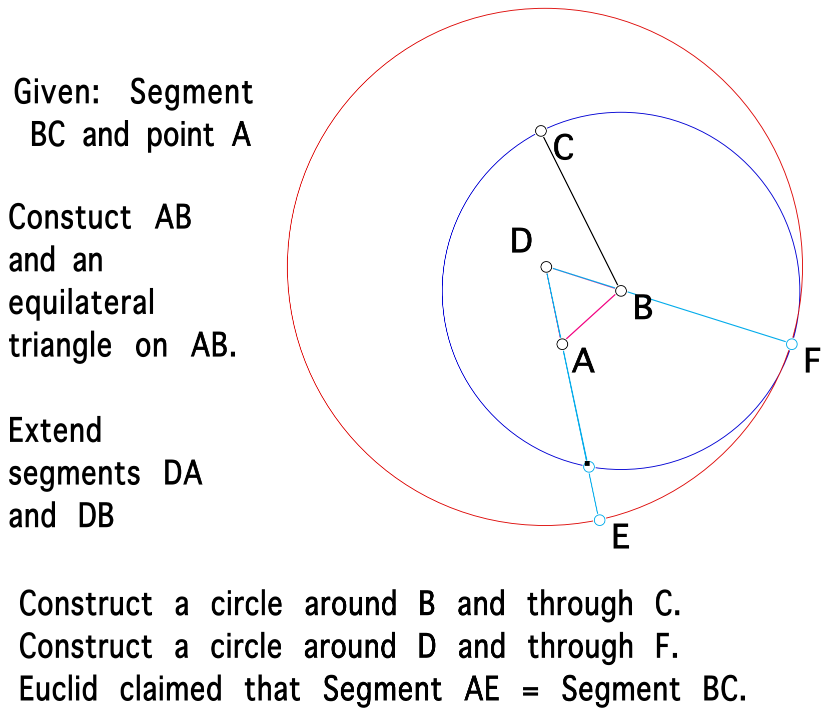


(Proclus)

Proclus argues that segment AB and AC were equal since points B and C lie upon the same circle, so segment AD would then be less than AC and AB. Similarly by construction, BD will be greater than BA. Therefore, since all of the segments are of different lengths, Triangle ABD is scalene as long as point D remains outside of the circle with center at B.

This poses a restriction upon which scalene triangles can be constructed by this method but it still gives a variety of scalene triangles and is therefore worth considering. Proclus' way of looking at the things that Euclid included, as well as excluded, is a true testament to insight into geometry and deductive logic. His assortment of alternative proofs and constructions, combined with his development of various cases that Euclid failed to address, make for an interesting discussion of geometry.

In some instances, it is his demonstrations of the various cases that is of great interest rather than the commentary or alternative proofs. Such is the case in his handling of Proposition 2 of the Elements which is " At any given point, construct a straight line equal to a given straight line." The construction that Euclid proposed is below:



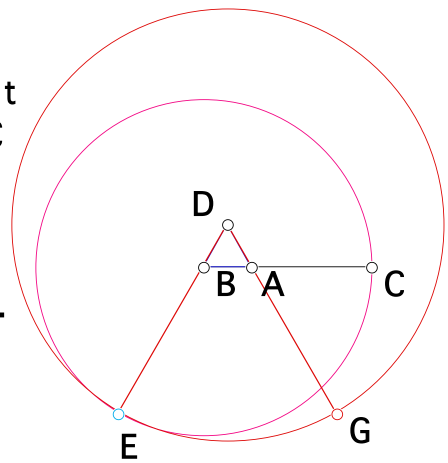
(Euclid)

Euclid's argument was that Segment BC = BF since they are enclosed in the same circle. Segment DB = DA by construction of equilateral triangles (which is the reason that Euclid began his elements with the equilateral method) . The third equivalence that Euclid addresses is that DF = DE since they lie in the same circle. With these three equivalencies, it is easy to see in the diagram above that if DA = DB and DE = DF, then AE = BF. And since BF = BC , then AE also equals BC which was what we wanted to show.

This construction for a so-called "Collapsible Compass" is one that Proclus put to the test by experimenting with different cases to show that this procedure always works. One case that Proclus tried was to let point A fall on segment BC, which requires a variation on the construction. He is quick to point out that Euclid never states that each case will require variations on construction, which could be confusing to an amateur geometer. For the case in which point A falls on Segment BC, Proclus shows us that a slightly different construction is needed.

Given:
Segment
BC
and

Point
A.



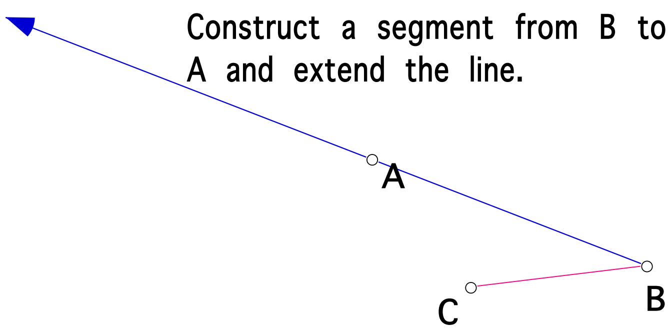
In attempting to use Propostion 2 to solve the case where point A falls on Segment BC, we see that it is necessary to use segment AC to show that AG is congruent to the original segment BC. In Euclid's construction, the segment AC never exists.

Since variations on the construction are needed to solve each case (including the case when BC is smaller than segment AB), the Elements should include either a demonstration of all cases or an alternative construction for Proposition II. In the interest of honoring Euclid's belief that we should learn for the sake of learning, I propose that we develop a new construction that does not require variations of construction depending upon where in the plane BC and point A lie. So, below is an alternative construction for Proposition II:

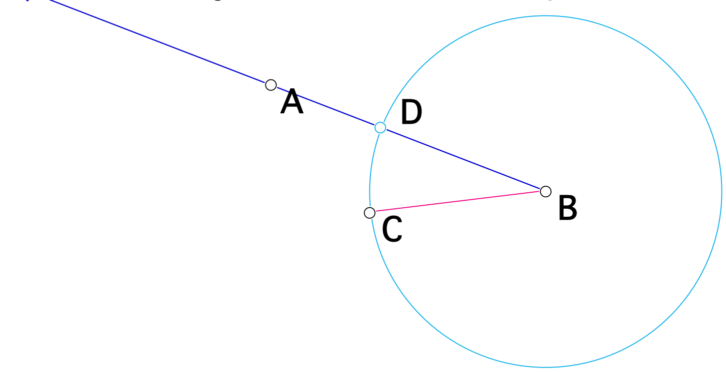
Given: Point A
and Segment BC

Place
distance BC
onto point A:

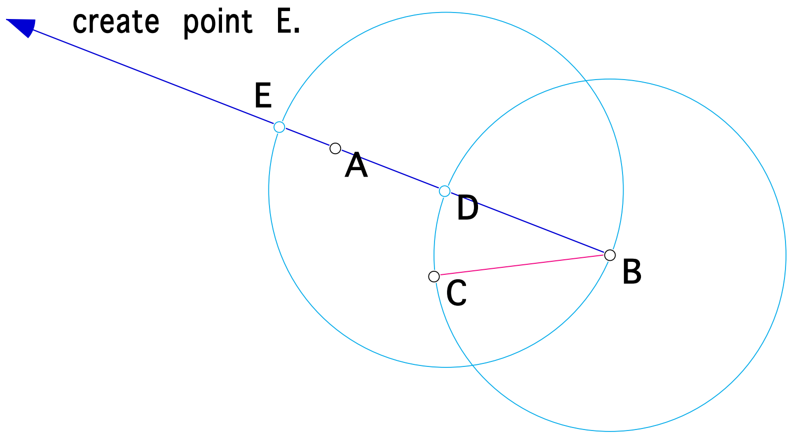
Construct a segment from B to A and extend the line.



Construct a circle around B and through point C to mark the distance BC on segment AB and create point D:

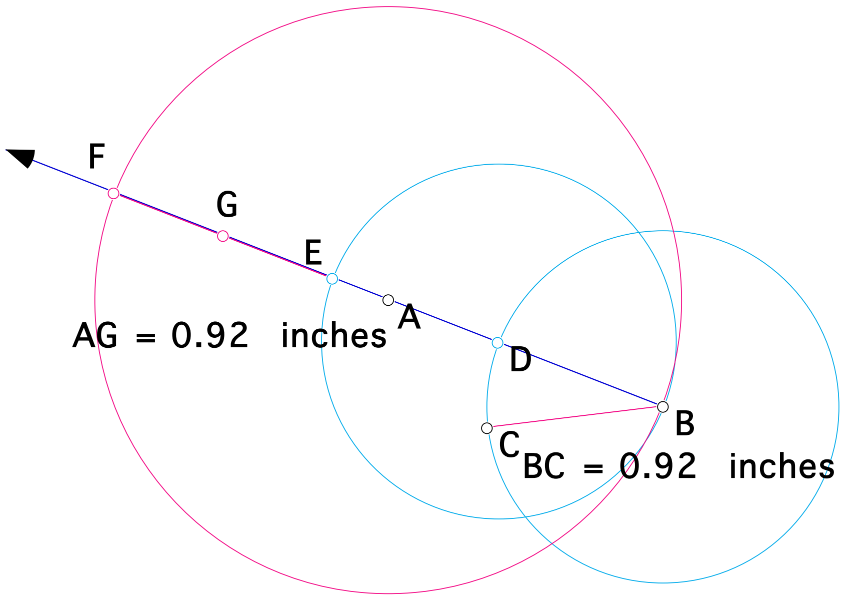


Construct a circle around D and through B which marks the distance BC again and will create point E.



Construct a circle around A and through point B which will create a new point F. It is then necessary to take the midpoint of the segment EF to create point G. The claim is that Segment AG will always be equal to the original Segment BC.

The final result will always be that AG = BC regardless of where the points are in the plane since the procedure does not vary with the each case. For example one case is that BC is less than the segment AB and another case is that AB is greater than the original BC.



Even though the construction always stay the same for each case, the proof of each case requires variances in the method. Just for the sake of showing that one proof is insufficient, there are more than 4 cases that should be shown before we can fully conclude the procedure. For the sake of time, we will only go through the proof for the case that is seen above in which AB is greater than CB and falls between ED.

We label the measure of BC as the letter "a"

which means that also DB = a

and ED = a

We label the measure of AD as the letter "b" . We can see from the construction that if DB = a and AD = b, then AB = (a + b).

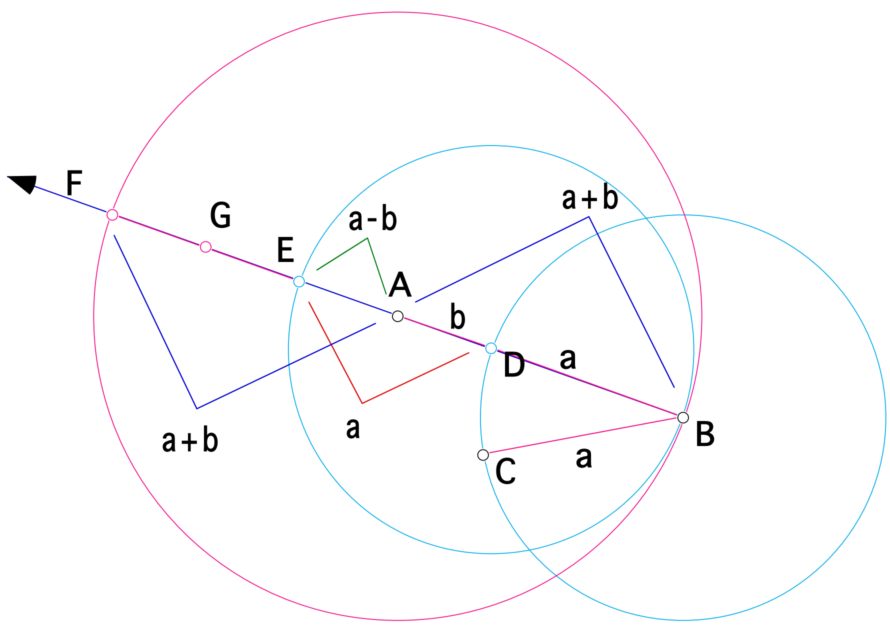
We can also see by construction that if ED = a and AD = b, then

the measure of EA = (a - b)

We can also see that Segment AF = (a + b) because
Segment AB = (a + b) as easily seen.
And since Segment AF and AB lie within the same circle, they are equal to each other.

To see that the midpoint of EF is the point that we seek, we notice that

$$EF = (a + b) - (a - b) = 2b.$$



Knowing this, it is easy to see that to place length "a" onto point A, we will need to use the fact that EF = 2b.
We can see from the construction that we already have (a-b) onto point A.
To get the measure (a-b) to be just "a", it would be necessary to add a length "b" onto it.

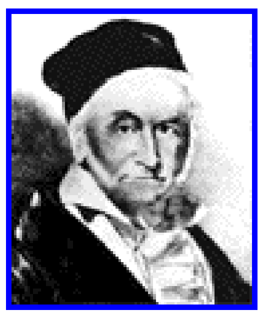
And since EF = 2b, it is easy to get a length of just b by taking the midpoint of EF.
The point that is formed is point G and therefore, Segment AG = a
and Segment BC = a,
Therefore, Segment AG = BC (which was the original length)
End of Proof.

With this construction, once one knows the procedure, it is not necessary to deal with the variations that occur due to the position of the givens. Euclid, apparently did not feel as though it was necessary to address the other cases, since some of the work must be left up to the student studying the Elements. In this day and age, students have a tendency to take advantage of the fact that we can merely lift our compass from the page and place it down somewhere else in the plane. However, Euclid refused to exclude the Collapsible Compass, perhaps because he believed that if something can't be proven, then it must not be true. Regardless, it is our duty as geometers to analyze the 'elements' of geometry and to develop new and innovative ways to handle abstract problems.

In fact, this kind of analysis of the Elements led mathematicians to develop an entirely new branch of geometry (Non-Euclidean Geometry). If Gauss, Lobachevsky, or Bolyai had not attempted to explore the idea that there only exists one line through a given point that is parallel to a given line, then the so-called 'Non-Euclidean' geometry may have never been developed.

Janos Bolyai

Johann Gauss

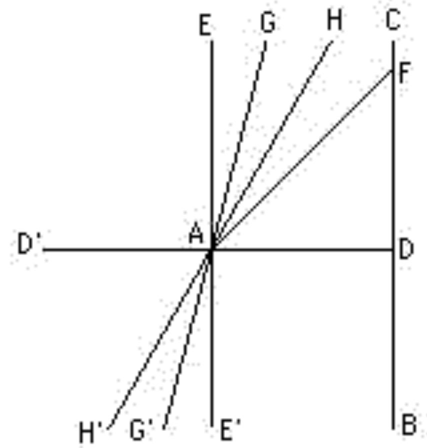


Nikolai Lobachevsky

This entire new system of mathematics was a direct result of a closer analysis of the Euclid's Postulate V, which claims that only one line parallel to a given line can pass through a fixed point given external to the line. All three of these mathematicians, independently, argued that all straight lines which in a plane go out from a point can, with reference to a given line in the same plane, be divided into two classes - into cutting and non-cutting. The boundary lines of the one and the other class of those lines will be called parallel to the given line. For this to be the case, an alternative postulate is necessary:

There exist two lines parallel to a given line through a given point not on the line.

The mathematics involved are too involved to discuss here, but the following is a summary of Lobachevsky's argument for the alternative postulate above.



AD is the perpendicular from A to BC.
AE is perpendicular to AD.
Within the angle EAD, some lines (such as AF) will meet BC.
Assume that AE is not the only line which does not meet BC, so let AG be another such line.
AF is a cutting line and AG is a non-cutting line.
There must be a boundary between cutting and non-cutting lines and we may take AH as this boundary.

Intense analysis of Euclid's Elements can open doors in mathematics that we never dreamed existed. Euclid's summary of geometry has possibly been the target of more criticism than any other text in history. However, by merely trying to point out the problems with the Elements, mathematicians have come to a better understanding of geometry - whether it be through alternative proofs or pointing out inconsistencies in the development. The entire history of mathematics would simply not have been the same if Euclid had not taken the time to organized geometry in a concise and understandable form. The knowledge that we can acquire from this text is not yet exhausted and probably never will be. Euclid has done all of us a great service in writing the Elements and it is up to all of us to learn from his great effort to help us do just that ------ learn!